

A time-dependent “Cosmological Constant” Fenomenology

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Abstract

We construct a cosmological toy model in which a step-function “cosmological constant” is taken into consideration beside ordinary matter. We assume that Λ takes two values depending on the epoch, and matter goes from a radiation dominated era to a dust dominated era. The model is exactly solvable and it can be compared with recent observations.

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1 Introduction

A cosmological constant term in the energy density of the universe turns out to be taken in a very serious consideration in the today research. Many experimental data on the structure of the present universe are compatible with models in which the cosmological constant term contribution in the density parameter Ω (the mean mass density relative to the critical density of the corresponding Einstein-de Sitter model $\rho_c = \frac{3H^2}{8\pi G}$) is relevant with respect to the matter term. This happens in the number count of galaxies as well as in the spectrum of the Cosmic Microwave Background Radiation (CMBR) [1], [2]. On one side there is the well known problem concerning the theoretical and experimental limits on the values the cosmological constant should assume, in the sense that they differ for 120 order of magnitude [3], [4]. On the other side, despite of these discrepancies, the presence of a cosmological constant term is requested to avoid the problems which are found comparing the age of the universe as coming from estimates on globular clusters with the value obtained from a standard model with an unitary density parameter, if one wants to give to the Hubble parameter H_o a value in agreement with the most recent results [2], [5], [6]. As it is well known, the value for the age comes out to be too low, while with a cosmological constant term one is able to find the agreement with the observations. Finally from the statistic of the phenomenon of the gravitational lensing it comes out an upper limit on Λ , that is its contribution to Ω seems to be less than 95% [7].

Furthermore the models in which a cosmological constant is present give rise to the inflationary epoch which solves the problems of the cosmological standard model such as the horizon, flatness, entropy problems [8], [9]. In such a context, the present value of the density parameter turns out to be very close to unity. If we consider that the observational estimates of Ω from the smaller scales of galaxies to scales of order of 10 *Mpc* give $0.05 \leq \Omega \leq 0.2$ [10], we come out at once to the very well known problem of Ω .

In this article we analyse some phenomenological aspects relatively to the presence of a cosmological constant in connection with the problem of Ω in the context of an exactly solved model. We consider a homogeneous and isotropic flat model with an inflationary epoch, a radiation dominated epoch and a matter dominated epoch, in which there is a residual cosmological constant. That is we consider the system of equations

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = \Lambda - p_m \quad (1)$$

$$\dot{\rho}_m + 3\frac{\dot{a}}{a}(p_m + \rho_m) = 0 \quad (2)$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3} + \frac{\rho_m}{3} \quad (3)$$

in which $a(t)$ is the expansion parameter; actually Λ is considered function of time, more precisely, as it will be shown, it will be considered piecewise constant; ρ_m , p_m are

the energy density and the pressure relative to the matter. The eqs. (1), (3) are the Einstein equations, while (2) is the contracted Bianchi identity. We are using units in which $8\pi G = c = 1$. Finally, considering standard matter, its state equation is, as usual, $p_m = (\gamma - 1)\rho_m$, with $1 \leq \gamma \leq 2$.

2 The Model

We describe three epochs in which we assume Λ and γ as step-functions, that is

$$\Lambda(t) = \begin{cases} \Lambda_1, & t_i < t < t_f \\ \Lambda_2, & t_f < t \end{cases} \quad (4)$$

$$\gamma(t) = \begin{cases} \gamma_1 = \frac{4}{3}, & t < t_{eq} \\ \gamma_2 = 1, & t_{eq} < t \end{cases} \quad (5)$$

in which Λ_1 is the value of the cosmological constant during the inflationary epoch, Λ_2 is the residual value, t_i , t_f are the initial and final times of the inflationary epoch, t_{eq} is the instant of equivalence between radiation and matter density. We solve the (1)-(3) for $t_i < t < t_f$, $t_f < t < t_{eq}$, $t_{eq} < t$ and then impose the continuity of the expansion parameter a and of the total energy density $\rho_{tot} = \Lambda + \rho_m$ in t_f , t_{eq} ; from (1)-(3) these conditions imply that the Hubble parameter $H = \frac{\dot{a}}{a}$ and the total density parameter $\Omega_{tot} = \frac{\rho_{tot}}{3H^2}$ are continuos at any t .

To solve (1)-(3), taking into account the state equation, we follow this way: we multiply eq. (3) by a factor β_i , then add it to eq. (1), obtaining

$$2\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) + (\beta_i + 3)\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{3}(\beta_i + 3)\Lambda_i - \frac{1}{3}[\beta_i - 3(\gamma_i - 1)]\rho_m = 0 \quad (6)$$

in which the index i takes into account the different values of γ and Λ in the different epochs, according to (4), (5). Taking $\beta_i = 3(\gamma_i - 1)$, we obtain a second order equation for $a(t)$ in which it does not appear explicetly the term relative to ρ_m . Eq. (6) is a Riccati-type equation in $\frac{\dot{a}}{a}$, which is possible to solve and one finds

$$a = a_\alpha \left[e^{\frac{3\gamma_i}{2}\sqrt{\frac{\Lambda_i}{3}}t} - c_\alpha e^{-\frac{3\gamma_i}{2}\sqrt{\frac{\Lambda_i}{3}}t} \right]^{2/3\gamma_i}. \quad (7)$$

From the (contracted) Bianchi identity, eq. (2), and from the (0,0) Einstein eq. (3) we get $c_\alpha = \frac{M_\alpha}{4\Lambda_i a_\alpha^{3\gamma_i}}$, where M_α is given by $M_\alpha = \rho_m a^{3\gamma_i}$.

It is interesting to note that the expression (7) for $a(t)$ presents a singularity for any (α, i) if $c_\alpha > 0$. This condition is verified, being c_α connected with the energy density of

matter. Thus there will be a t_s such that $a(t_s) = 0$ and it seems natural to put the time origin in t_s . Redefining t such that $a|_{t=0} = 0$, the solution takes the form

$$a = \begin{cases} a_1 \left[e^{2\sqrt{\frac{\Lambda_1}{3}} t} - e^{-2\sqrt{\frac{\Lambda_1}{3}} t} \right]^{1/2}, & t_i < t < t_f, & \frac{M_1}{4 \Lambda_1 a_1^4} = 1; \\ a_2 \left[e^{2\sqrt{\frac{\Lambda_2}{3}} t} - c_2 e^{-2\sqrt{\frac{\Lambda_2}{3}} t} \right]^{1/2}, & t_f < t < t_{eq}, & c_2 = \frac{M_2}{4 \Lambda_2 a_2^4}; \\ a_3 \left[e^{\frac{3}{2}\sqrt{\frac{\Lambda_2}{3}} t} - c_3 e^{-\frac{3}{2}\sqrt{\frac{\Lambda_2}{3}} t} \right]^{2/3}, & t_{eq} < t, & c_3 = \frac{M_3}{4 \Lambda_2 a_3^3}; \end{cases} \quad (8)$$

in which a_α, c_α are not the same as given in (7) but they have been opportunely redefined, because of the fixing of the time beginning. Of course, the meaning of the times t_i, t_f, t_{eq} of the (4), (5) coincide with those of (8). For the total energy density one has the expression for the different epochs:

$$\rho_{tot} = \begin{cases} \Lambda_1 + \frac{M_1}{a^4}, & t_i < t < t_f \\ \Lambda_2 + \frac{M_2}{a^4}, & t_f < t < t_{eq} \\ \Lambda_2 + \frac{M_3}{a^3}, & t_{eq} < t \end{cases} \quad (9)$$

As we have said above, we impose the continuity conditions

$$a|_{t_f^-} = a|_{t_f^+}, \quad \rho_{tot}|_{t_f^-} = \rho_{tot}|_{t_f^+} \quad (10)$$

$$a|_{t_{eq}^-} = a|_{t_{eq}^+}, \quad \rho_{tot}|_{t_{eq}^-} = \rho_{tot}|_{t_{eq}^+}. \quad (11)$$

In this way, from (3) and from the definition of Ω , we have that $H(t)$ and $\Omega(t)$ are continuos at any t .

It is noteworthy that, assuming (4), (5) and taking into account (9), we introduce a discontinuity, at the instants t_f, t_{eq} , in the equation of state [11] for the total energy density $p_{tot} = (\gamma_{tot} - 1) \rho_{tot}$, being $\gamma_{tot} = \frac{\gamma(t) \rho_m}{\Lambda(t) + \rho_m}$; then the entropy S and the scalar curvature R are discontinuous too in the same instants, being respectively

$$S = \frac{a^3}{T} \gamma_{tot}; \quad R = -6 \left(\frac{\ddot{a}}{a} + H^2 \right), \quad (12)$$

and considering that (1) can be written as

$$2 \left(\frac{\ddot{a}}{a} \right) + \left(\frac{\dot{a}}{a} \right)^2 = -(\gamma_{tot} - 1) \rho_{tot}. \quad (13)$$

Thus the entropy production in t_f takes place through a phase transition and takes into account the production of matter at the end of the inflationary epoch through the condition (10) on ρ_{tot} .

Referring to the first of (9), we can consider the beginning of inflationary epoch t_i as the instant in which there is equivalence between the energy density relative to the cosmological constant and the one relative to the matter, that is

$$\Lambda_1 = \frac{M_1}{a^4} \Big|_{t_i} \quad (14)$$

so that, for $t \ll t_i$, one would have a radiation dominated pre-inflation epoch and for $t \gg t_i$ one has the inflationary epoch dominated by the cosmological constant. Thus

$$t_i = \frac{1}{2} \sqrt{\frac{3}{\Lambda_1}} \ln(1 + \sqrt{2}). \quad (15)$$

To solve the problems of the standard model (horizon, flatness, entropy), the number of e-folding during inflation has to be $N_{e-folding} \geq 67$ [8], [9]. From this condition assuming that $\sqrt{\Lambda_1/3} t \gg 1$ during inflation, which seems to be very reasonably, we get a condition on Λ_1 and t_f , given by $\Lambda_1 t_f^2 \geq 1.4 \cdot 10^4$. The validity of the model is assumed, of course, for $t \geq t_P$, with $t_P \simeq 5.4 \cdot 10^{-44}$ s the Planckian time. Imposing that at t_P the total energy density is just the Planck density, we obtain an estimate for Λ_1 . Thus we write

$$\frac{M_1}{a^4} \Big|_{t_P} = \rho_P \quad (16)$$

in which ρ_P is the Planckian density. If we assume that at t_P the radiation dominates, that is $t_P \ll t_i$, and we can develop the first of (7) in $\sqrt{\Lambda_1/3} t$ to the first order (radiation behavior for $a(t)$), obtaining at once an estimates for Λ_1 , which comes out to be negative. This means that we have to take into account also the cosmological constant term. Assuming in the first approximation that it dominates, we get a lower bound for Λ_1 , that is $\Lambda_1 \geq 8.4 \cdot 10^{87} \text{ s}^{-2}$. From (15) one gets $t_i \leq 8.3 \cdot 10^{-45} \text{ s}$, which is less than the Planckian time. This only means that our model can be considered reasonable starting with a cosmological constant dominated era.

The continuity conditions (10), (11), give some relations between the constant a_α , c_α present in eq. (8); taking into account the relation between Λ_1 , t_f we get

$$a_1 = \left[\frac{2}{1 + \sqrt{\Lambda_{12}}} \right]^{1/2} \frac{a_o e^{\sqrt{\Lambda_2/3} (t_f - t_o)} [1 - r_{12} e^{4\sqrt{\Lambda_2/3} (t_f - t_{eq})}]^{1/6}}{e^{\sqrt{\Lambda_1/3} t_f} [1 - r_{12} e^{\sqrt{\Lambda_2/3} (4t_f - t_{eq} - 3t_o)}]^{2/3}} \quad (17)$$

$$a_2 = \frac{a_o [1 - r_{12} e^{4\sqrt{\Lambda_2/3} (t_f - t_{eq})}]^{1/6}}{e^{\sqrt{\Lambda_2/3} t_o} [1 - r_{12} e^{\sqrt{\Lambda_2/3} (4t_f - t_{eq} - 3t_o)}]^{2/3}} \quad (18)$$

$$a_3 = \frac{a_o}{e^{\sqrt{\Lambda_2/3} t_o} [1 - r_{12} e^{\sqrt{\Lambda_2/3} (4t_f - t_{eq} - 3t_o)}]^{2/3}} \quad (19)$$

$$c_2 = r_{12} e^{4\sqrt{\Lambda_2/3} t_f}, \quad c_3 = r_{12} e^{\sqrt{\Lambda_2/3} (4t_f - t_{eq})} \quad (20)$$

in which t_o is the present age, $a_o = a|_{t_o}$, $\Lambda_{12} = \sqrt{\frac{\Lambda_1}{\Lambda_2}}$ and $r_{12} \simeq \frac{-1 + \sqrt{\frac{\Lambda_1}{\Lambda_2}}}{1 + \sqrt{\frac{\Lambda_1}{\Lambda_2}}}$.

A very reasonable assumption is $\Lambda_1 > \Lambda_2$; it can be seen that this is equivalent to have a_1, a_2, a_3 real and c_2, c_3 positive.

3 Compatibility with observations

The comparison with the experimental data can be done through the constants M_2, M_3 , which can be connected with the energy density of the matter and of the radiation at the present age, which we call respectively μ_o, ε_o .

Assuming $\varepsilon_o \ll \mu_o$, one can write

$$M_2 = \varepsilon_o a_o^4, \quad M_3 = \mu_o a_o^3. \quad (21)$$

Using the relations existing between c_2, M_2 and c_3, M_3 as given by (8) and taking into account eqs. (17)-(20), we get

$$\varepsilon_o = \frac{4 \Lambda_2 c_2 a_o^4}{a_o^4} = \frac{4 \Lambda_2 r_{12} e^{\sqrt{\Lambda_2/3} (t_f - t_o)} [1 - r_{12} e^{4\sqrt{\Lambda_2/3} (t_f - t_{eq})}]^{2/3}}{[1 - r_{12} e^{\sqrt{\Lambda_2/3} (4t_f - t_{eq} - 3t_o)}]^{8/3}} \quad (22)$$

$$\mu_o = \frac{4 \Lambda_2 c_3 a_o^3}{a_o^3} = \frac{4 \Lambda_2 r_{12} e^{\sqrt{\Lambda_2/3} (4t_f - t_{eq} - 3t_o)}}{[1 - r_{12} e^{\sqrt{\Lambda_2/3} (4t_f - t_{eq} - 3t_o)}]^2}. \quad (23)$$

Being $t_f, t_{eq} \ll t_o$ we also get an expression for the present age

$$t_o = \frac{1}{H_o} \int_1^\infty \frac{dx}{x \sqrt{\Omega_{\mu_o} x^3 + \Omega_{\Lambda_o}}} \quad (24)$$

in which $H_o, \Omega_{\mu_o}, \Omega_{\Lambda_o}$ are respectively the Hubble parameter, the contribution of matter and cosmological constant to Ω at the present age. The observational value of ε_o comes from the black body law

$$\varepsilon_o = \sigma_B T_o^4 \simeq 4.649 \cdot 10^{-34} \text{ g cm}^{-3} \quad (25)$$

in which σ_B is the radiation-density constant and $T_o = 2.726 \pm 0.010 \text{ } ^\circ K$ is the CMBR temperature [12].

The most recent estimates of Hubble parameter H_o come from various distances calibrators on different scales and are (see [2], [5], [6], [13])

$$H_o = 100 h \text{ Km s}^{-1} \text{ Mpc}^{-1}, \quad 0.55 \leq h \leq 0.85. \quad (26)$$

The most recent estimates of Ω_{μ_o} come from comparison of CDM models vs. redshift surveys and from studies on the dynamics of cosmic flows; they give (see [14], [15], [16]) $\Omega_{\mu_o} \geq 0.2 \div 0.3$ at 2σ level or more. Thus, from the definition of Ω , being the model flat, we get

$$\mu_{o, \text{ est. }} = 3 H_o^2 \Omega_{\mu_o} = 1.879 \cdot 10^{-29} h^2 \Omega_{\mu_o} g \text{ cm}^{-3} \quad (27)$$

$$\Lambda_{2, \text{ est. }} = 3 H_o^2 \Omega_{\Lambda_o} = 3.151 \cdot 10^{-35} h^2 \Omega_{\Lambda_o} s^{-2}. \quad (28)$$

Comparing (27) with (25) we see that the assumption $\varepsilon_o \ll \mu_o$ is justified.

We obtain an estimate of t_f considering that, for $t_f < t < t_{eq}$, the matter is prevalently ultrarelativistic, thus one has the relation between the energy density ρ_m and the absolute temperature T

$$\rho_m = \frac{\sigma_B}{2} g(T) T^4 \quad (29)$$

in which $g(T)$ are the effective spin degree of freedom. Considering the epoch just after t_f , the radiation dominates on the cosmological constant and, being verified by construction the hypothesis of an efficient reheating, equating the expression of $\rho_m(t_f^+)$ with $\rho_m(T_f^+)$, one gets t_f in terms of T_f^+ , that is, the right limit on T_f . Such a quantity is constrained by the baryogenesis [17], being $T_f^+ \geq 10^{10} \text{ GeV}$, which implies a constraint on t_f , that is $t_f \leq 3.7 \cdot 10^{-29} \text{ s}$ and this is compatible with the relation between Λ_1 , t_f and with the constraint we found on Λ_1 .

Giving to t_f , t_{eq} , Ω_{μ_o} respectively the values $t_f = 10^{-29}$, $t_{eq} = 10^{12}$, $\Omega_{\mu_o} = 0.3$, compatible with all the considerations we have done, we find that the values of μ_o , ε_o given from the model are substantially compatible with those coming from the observations.

In particular we find that the values of $\varepsilon_{o, \text{ mod. }}$ obtained from the model, that is from (22), differ from the values $\varepsilon_{o, \text{ est. }}$ given from (25) when h varies according with (26) as

$$\left. \frac{\varepsilon_{o, \text{ est. }} - \varepsilon_{o, \text{ mod. }}}{\varepsilon_{o, \text{ est. }}} \right|_{h=0.55} = 0.43, \quad \left. \frac{\varepsilon_{o, \text{ est. }} - \varepsilon_{o, \text{ mod. }}}{\varepsilon_{o, \text{ est. }}} \right|_{h=0.85} = -0.83, \quad (30)$$

essentially unchanged for T_o in the experimental errors and for increasing Ω_{μ_o} . This means that there is full compatibility; the value of h which minimizes the square of the relative difference for ε_o to a value significantly lower than 10^{-4} is given by $h = 0.6777$, which is compatible with the estimate given by (26).

The values of $\mu_{o, \text{ mod. }}$ obtained from the model, that is from (23), differ from the values $\mu_{o, \text{ est. }}$ given from (27) when h varies according with (26) as

$$\left. \frac{\mu_{o, \text{ est. }} - \mu_{o, \text{ mod. }}}{\mu_{o, \text{ est. }}} \right|_{h=0.55} = 1.8 \cdot 10^{-6}, \quad \left. \frac{\mu_{o, \text{ est. }} - \mu_{o, \text{ mod. }}}{\mu_{o, \text{ est. }}} \right|_{h=0.85} = 2.2 \cdot 10^{-6}. \quad (31)$$

In this case we don't find a full compatibility but the relative difference is of one part over 10^6 . The agreement increases for increasing Ω_{μ_o} and decreasing h .

The values of the equivalence temperature, for $h = 0.6777$, are found to be $T_{eq}|_{T_o=2.716} = 1.303 \text{ eV}$, $T_{eq}|_{T_o=2.736} = 1.313 \text{ eV}$, which are just one order of magnitude less than the

decoupling temperature, given by $T_{dec} = 13 \text{ eV}$, and for the temperature immediately after inflation $T_f^+|_{T_o=2.716} = 5.041 \cdot 10^{18} \text{ GeV}$, $T_f^+|_{T_o=2.736} = 5.078 \cdot 10^{18} \text{ GeV}$, which are compatible with the constraint imposed by the baryogenesis.

Moreover we find for the present age the value $t_{o, mod.} = 13.9 \cdot 10^9 \text{ y}$, for $h = 0.6777$, compatible with the most recent estimates which give $t_{o, est.} = 14 \pm 2 \cdot 10^9 \text{ y}$ [6]. Of course the agreement decreases with increasing h and increases with decreases $\Omega_{\mu o}$.

Finally we want to stress that the behavior of $a(t)$ given from (8), seems to be quite different from the standard expansion $a \propto t^{2/3}$ of an universe made of dust, as our present universe appears to be. One gets the standard expansion from (8) if $\sqrt{\Lambda_2/3} t \ll 1$; this turns out to be no longer strictly verified at the present age, since we get $\sqrt{\Lambda_2/3} t_o \simeq 0.7$. This means that the effects of cosmological constant term on the expansion are no longer negligible. This is something which could be taken into consideration in the experimental measurements.

4 Conclusions

We have constructed a phenomenological model in which the cosmological constant is considered as a step function depending on the epoch. Also ordinary standard matter is taken into consideration as radiation at the beginning and as dust after the equivalence. The model is exactly solvable and allows to implement an inflationary epoch after which we found substantial agreement with the most recent observational data concerning the values of Ω , the age of the universe, the CMBR temperature at equivalence and today. The construction can be perfectly compatible with such models which call for an amount of barionic matter, cold dark matter and cosmological constant in order to explain cosmological dynamics and large scale structure formation after an inflationary expansion [18]. We have to remark that this is a toy model in which a sharp transition between the two values of Λ is invoked but it justifies how a time-dependent cosmological constant could affect early and present cosmological dynamics and, in some sense, be in agreement with observational data. That is this analysis confirms how important it could be to specify the concept of “cosmological constant” in a wider way and in a more general context, such as, for example, that of the nonminimally coupled scalar-tensor theories, where also scalar field(s) can be considered in dynamics and, in general, be nonminimally coupled with geometry [19].

It is noteworthy that the technique we used to solve (1)-(3) of multiplying the first order Einstein equation for an opportune factor and adding what obtained to the second order Einstein equation to get an equation in which it does not appear explicitly the term in ρ_m can be used also in more general cases where a scalar field energy density is present together with that of matter. The case we have considered is a particular case of constant potential and zero initial conditions for the scalar field dynamics.

Furthermore we have to say that the problem of “graceful exit” has to be taken into consideration; that is we have to consider what kind of realistic inflationary model could

implement the above dynamics and allow the requested Λ transition.

As concluding remark we consider with more attention the non completely satisfactory compatibility expressed by (31); actually, giving to h the value $h = 0.7$ compatible with the data, the square of the relative difference reaches its minimum for a negative value of t_{eq} . Thus the transition from radiation to matter would occur before the initial singularity, which means that to have full compatibility relatively to the matter energy density, a radiation dominated era does not find space. This suggests an explanation of the disagreement concerning the matter in presence of radiation, because we have totally neglected its presence after t_{eq} but we have taken it into account in the comparison with the data. The order of magnitude of the disagreement is in fact less than the order of magnitude of the ratio between the present energy density of the radiation and that of the matter (see eqs. (31) and (25), (27)). It is reasonable to think, in order to solve these discrepancies, that a model with non zero initial conditions on the scalar field (see [20]) can be revisited with the phenomenological approach used in this paper. This will be one of the further developments of our future research.

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